The Centre of Theoretical and Mathematical Physics at the University of Cape Town announces the Jack de Wet Student Competition 2011. The Department of Physics and the Department of Mathematics at the University of Cape Town sponsor the competition with R2000 to be awarded for the best and most elegant solution to the problem set out below.

Science students up to and including Masters Level who are registered at the University of Cape Town, the University of Stellenbosch, the University of the Western Cape or the African Institute for Mathematical Sciences are eligible to participate. Contributions must be submitted by 4pm Wednesday, 15 February 2012, to any member of the prize committee: Profs. Barashenkov, Dominguez, Ianovsky, Peshier, Tupper, Viollier, Weigert and Weltman. Contact information can be found on the home pages of the Department of Physics and the Department of Mathematics at the University of Cape Town. We require the entrant to provide his or her study record to allow us to verify academic affiliation. Candidates have to attest that their contributions represent own work.

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**The beauty of spin chains: from magnetism to string theory**

: Set by Dr. Jeff Murugan, MAM, UCT :

"Those who are not shocked when they first come across quantum theory cannot possibly have understood it." - Niels Bohr

Not long after its initial formulation, it was realized that Quantum Mechanics held the key to explaining a number of well-known and, at that stage, difficult problems. In particular, the quantum mechanical exchange interaction between electron spins in neighbouring atoms with overlapping orbital wavefunctions, due to a combination of Coulomb forces and Pauli exclusion provided the first real theory of ferromagnetism. Today, this so-called XXX Heisenberg spin-chain is not only one of the best models of a ferromagnet we have but has also surfaced in a number of other areas. For example, it plays a crucial role in understanding the integrability properties of a class of supersymmetric non-Abelian gauge theories thought to be dual to superstring theory. The dynamical variable in this problem is the spin vector

![Figure 1: From left to right: The 1-dimensional Heisenberg ferromagnet; its one magnon excitation; the anti-ferromagnet and its one magnon excitation.](image-url)
\( \vec{S}_n = (S^x_n, S^y_n, S^z_n) \) with quantum number \( s = 1/2 \) on a 1-dimensional lattice of \( N \) sites. For concreteness sake, let’s consider only closed spin-chains (see Figure.1). For such a chain of spins, periodic boundary conditions require that \( \vec{S}_{N+1} = \vec{S}_1 \). If we assume that each spin interacts only with its nearest neighbour then the dynamics of the spin-chain is defined through its Hamiltonian,

\[
H = -J \sum_{n=1}^{N} \vec{S}_n \cdot \vec{S}_{n+1}
\]  

1. The algebra of spin operators:

- Give an interpretation of the positive constant \( J \) in the above Hamiltonian.
- Rewrite the Hamiltonian above in terms of the operators \( S^\pm_n \equiv S^x_n \pm iS^y_n \), and \( S^z_n \)
- The Hamiltonian \( H \) acts on a \( 2N \)-dimensional Hilbert space spanned by the orthogonal basis \( |\sigma_1 \cdots \sigma_N\rangle \) where \( \sigma_k = \uparrow \) and \( \sigma_k = \downarrow \) represent an up and down spin respectively at site \( k \). Work out the rules for the action of the spin operators on the basis vectors \( |\sigma_1 \cdots \sigma_N\rangle \) with \( \sigma_k = \uparrow, \downarrow \) and use these rules to determine the commutators \([S^z_n, S^\pm_n] \) and \([S^+_n, S^-_n] \) (in units of \( \hbar = 1 \)).

By now, you probably appreciate the fact that, at the end of the day, solving a quantum mechanics problem boils down to diagonalizing the Hamiltonian for the system to compute its eigenvectors. Once these are determined, any physical quantities of interest can be calculated by evaluating the expectation values of the corresponding operators. The problem is that, in general, this is a computationally difficult problem for any but the lowest values of \( N \). In the case of the Heisenberg spin-chain, the problem of diagonalizing the Hamiltonian matrix can be made a little simpler by using the results of your computation above to express \( H \) as a real, symmetric \( 2^N \times 2^N \) matrix and then rewriting it in a block diagonal form.

Fortunately (for you), Hans Bethe solved the problem exactly in 1931 already. His method, the so-called Bethe Ansatz, is a transformation of basis that does not have to be supplemented by any numerical diagonalization and works for arbitrary \( N \). However, in order to apply to the Heisenberg model, the Bethe ansatz does rely crucially on

1. Symmetries of the XXX spin-chain:

   - Show that the Hamiltonian (1) is invariant under (a) rotations about the \( z \)-axis in spin space and (b) discrete translations by any number of lattice sites. (Hint: for part (b), consider the translation operator \( \hat{T} \) whose action on, for example, the configuration \( |\uparrow\uparrow\downarrow\downarrow\rangle \) is \( \hat{T}|\uparrow\uparrow\downarrow\downarrow\rangle = |\uparrow\downarrow\uparrow\downarrow\rangle \))
   - Using this, or otherwise, show that the \( z \)-component of the total spin \( S^z_T = \sum_{n=1}^{N} S^z_n \) is conserved.
3. Now argue that organizing the basis vectors according to the quantum numbers $S_T^z = N/2 - r$ (where $r$ is the number of down spins in a lattice of up spins) is sufficient to block diagonalize $H$.

Before introducing the power of the Bethe ansatz, let’s first set some benchmarks by looking at the lowest lying excitations of the spin-chain.

- **Diagonalizing Hamiltonian blocks:**
  
  1. To begin with, the block with $r = 0$ has just one vector $|F\rangle \equiv |\uparrow\uparrow\cdots\uparrow\rangle$. Show that this vector is an eigenstate of the Hamiltonian $H$ and compute its energy $E_0$.

  2. The $r = 1$ block is an $N \times N$ matrix spanned by the $N$ basis vectors $|n\rangle = S^{-}_n |F\rangle$ ($n = 1 \ldots N$). Show that the translationally invariant basis

$$|\psi\rangle = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} e^{i kn} |n\rangle,$$

with wavenumbers $k = 2\pi m/N$, $m = 0, 1, \ldots N - 1$, diagonalizes the $r = 1$ block of $H$ by showing that $|\psi\rangle$ is an eigenvalue of the translation operator $\hat{T}$ as well as the Hamiltonian. Compute the eigenvalues in each case.

The vectors $|\psi\rangle$ constructed above represent the so-called *magnon* excitations of the spin-chain in which the perfectly aligned spins of the ferromagnetic ground state $|F\rangle$ are periodically disturbed by a spin-wave with wavelength $\lambda = 2\pi/k$. This sounds like we’ve made great progress but the problem is that for $r \geq 2$, even such symmetry-adapted bases (which code translational, rotational and even reflectional symmetries of the lattice) are insufficient to diagonalize the Hamiltonian matrix. This is where Bethe’s method shines! To understand how the Bethe ansatz works, let’s look at the $r = 1$ block again (since we already know the answer):

- **The Bethe ansatz for $r = 1$:**

  1. Write a general vector in the $r = 1$ subspace as a superposition of the basis vectors as

$$|\psi\rangle = \sum_{n=1}^{N} a(n) |n\rangle$$

and, by plugging this into the eigenvalue equation $H|\psi\rangle = E|\psi\rangle$, find the recursion relation satisfied by the coefficients $a(n)$.

  2. Solve this recursion relation for its $N$ linearly independent solutions and check that (after normalization) this agrees with eq.(2).
At this point, this does not seem like a big deal. But that’s like not being surprised when you discover that the new jackhammer you got can hammer a nail into a piece of wood! You really only appreciate it when you try to hammer into cement. In this case, it is the two-magnon sector of the theory.

- **The Bethe ansatz for** $r = 2$:

  1. In this invariant subspace, a general eigenvector can be written as
     \[
     |\psi\rangle = \sum_{1 \leq n_1 < n_2 \leq N} a(n_1, n_2)|n_1, n_2\rangle, \tag{4}
     \]
     where $|n_1, n_2\rangle = S^-_{n_1} S^-_{n_2}|F\rangle$. What is the dimension of this subspace?

  2. To solve for the coefficients $a(n_1, n_2)$, take as an ansatz
     \[
     a(n_1, n_2) = Ae^{i(k_1 n_1 + k_2 n_2)} + A'e^{i(k_1 n_1 + k_2 n_2)}, \tag{5}
     \]
     and explain (with a physical interpretation) why you can’t set $A = A'$ and $k_1 = k_2$.

  3. With the ansatz above, show that the eigenvalue equation for $|\psi\rangle$ translates into the recursion relations
     \[
     2(E - E_0) a(n_1, n_2) = J [4a(n_1, n_2) - a(n_1 - 1, n_2) - a(n_1 + 1, n_2) - a(n_1, n_2 - 1) - a(n_1, n_2 + 1)], \tag{6}
     \]
     for $n_2 > n_1 + 1$ and
     \[
     2(E - E_0) a(n_1, n_2) = J [2a(n_1, n_2) - a(n_1 - 1, n_2) - a(n_1, n_2 + 1)],
     \]
     for $n_2 = n_1 + 1$. By subtracting the second equation from the first and setting $n_2 = n_1 + 1$, argue that the $a(n_1, n_2)$ are solutions of this set of linear relations if they (a) are of the form (5) and satisfy
     \[
     2a(n_1, n_1 + 1) = a(n_1, n_1) + a(n_1 + 1, n_1 + 1). \tag{7}
     \]

  4. Show that this requirement can be incorporated into the ansatz by adding in extra phase factors so that
     \[
     a(n_1, n_2) = e^{i(k_1 n_1 + k_2 n_2 + \theta/2)} + e^{i(k_1 n_1 + k_2 n_2 - \theta/2)}, \tag{6}
     \]
     where the phase angle $\theta$ depends on the, as yet undetermined momenta of the Bethe ansatz wavefunction, $k_1$ and $k_2$ via
     \[
     2 \cot \frac{\theta}{2} = \cot \frac{k_1}{2} - \cot \frac{k_2}{2}. \tag{7}
     \]
5. One final (formal) step in Bethe’s prescription is required to fix the momenta $k_1$ and $k_2$. This can be done by demanding that the wavefunction $|\psi\rangle$ be translationally invariant i.e. that $a(n_1, n_2) = a(n_2, n_1 + N)$. Show that this condition is equivalent to the statements that

$$Nk_1 = 2\pi\lambda_1 + \theta$$
$$Nk_2 = 2\pi\lambda_2 - \theta,$$

where the integers $\lambda_i \in \{0, 1, \ldots, N - 1\}$ are the Bethe quantum numbers.

To summarize then, in order to diagonalize the $r = 2$ block of the Hamiltonian, all that we need to do is to find all integer pairs of Bethe quantum numbers $(\lambda_1, \lambda_2)$ that solve eqs. (7) and (9). The truly neat thing about the Bethe ansatz is that once the $r = 2$ case is understood, generalization to $r > 2$ is straightforward. Try it!