The thermodynamics of dense non-commutative fermion gases

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4. Spectrum of the 2-D infinite spherical well
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8. Conclusions
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One expects that the effects of non-commutativity on the one particle level will be minute and not experimentally detectable.
Introduction and Motivation (contd)

For large numbers of particles the effects of non-commutativity may, however, be on a macroscopic scale. It therefore makes sense to investigate the thermodynamics of such systems to detect the physical consequences of non-commutativity and to investigate the appropriateness of different types of non-commutativity.
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Here we focus on two and three dimensional non-commutative Fermi gases in the high density limit, where the effects of non-commutativity are expected to show up.
Formulation of NC quantum mechanics

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Our first step will be to identify the non-commutative analogues of these.
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- Defining annihilation and creation operators $b = \frac{1}{\sqrt{2\theta}}(\hat{x} + i\hat{y})$, $b^\dagger = \frac{1}{\sqrt{2\theta}}(\hat{x} - i\hat{y})$ non-commutative configuration space, $\mathcal{H}_c$, is isomorphic to Fock space
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  $\mathcal{H}_q = \left\{ \psi(\hat{x}, \hat{y}) : \psi(\hat{x}, \hat{y}) \in B(\mathcal{H}_c) , \text{tr}_c(\psi^\dagger(\hat{x}, \hat{y})\psi(\hat{x}, \hat{y})) < \infty \right\}$
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- This space has a natural inner product and norm
  $(\phi(\hat{x}_1, \hat{x}_2), \psi(\hat{x}_1, \hat{x}_2)) = tr_c(\phi(\hat{x}_1, \hat{x}_2)^\dagger\psi(\hat{x}_1, \hat{x}_2))$. 
The next step in building the quantum system is to find a representation for the non-commutative Heisenberg algebra on $\mathcal{H}_q$. In two dimensions this reads

$$[x_i, p_j] = i\hbar \delta_{i,j}, \quad [x_i, x_j] = i\theta \epsilon_{i,j} \quad [p_i, p_j] = 0.$$
The next step in building the quantum system is to find a representation for the non-commutative Heisenberg algebra on $\mathcal{H}_q$. In two dimensions this reads

$$\begin{align*}
[x_i, p_j] &= i\hbar \delta_{i,j}, \\
[x_i, x_j] &= i\theta \epsilon_{i,j}, \\
[p_i, p_j] &= 0.
\end{align*}$$

A unitary representation of this algebra in terms of operators $\hat{X}_i$ and $\hat{P}_i$ acting on $\mathcal{H}_q$ is easily found to be

$$\begin{align*}
\hat{X}_i\psi(\hat{x}_1, \hat{x}_2) &= \hat{x}_i\psi(\hat{x}_1, \hat{x}_2), \\
\hat{P}_i\psi(\hat{x}_1, \hat{x}_2) &= \frac{\hbar}{\theta} \epsilon_{i,j}[\hat{x}_j, \psi(\hat{x}_1, \hat{x}_2)],
\end{align*}$$

i.e., the position acts by left multiplication and the momentum adjointly.
Interpretation of NC quantum mechanics

The interpretation is as in usually quantum mechanics with $\mathcal{H}_q$ representing the state space, i.e., physical observables are represented by hermitian operators on $\mathcal{H}_q$, a measurement yields an eigenvalue, $a$, with probability $\text{tr}(\rho \pi_a)$ with $\rho$ the density matrix and $\pi_a = |a\rangle \langle a|$ the projection on the eigenstate $|a\rangle$. 
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Position measurement is, however, different as we cannot construct simultaneous eigenstates of $\hat{X}_1$ and $\hat{X}_2$. However, we can give meaning to this in the sense of a weak measurement (POVM), based on minimal uncertainty states for position.
The Hamiltonian for the spherical well reads

\[ \hat{H} = \frac{P^2 \psi}{2\mu} + (V_1 P + V_2 Q). \]

with

\[ P = \sum_{n=0}^{M} |n\rangle\langle n|, \quad Q = \sum_{n=M+1}^{\infty} |n\rangle\langle n|. \]

The radius of the disc is given by \( R^2 = \theta(2M + 1). \)
Spectrum of the 2-D infinite spherical well

The energies of the infinite well for positive angular momentum is obtained as

\[ L^m_{M+1} \left( \frac{\theta k^2}{2} \right) = 0, \ m \geq 0, \quad k^2 = \frac{2\mu E}{\hbar^2}, \]

and for negative angular momentum as

\[ L^m_{M+m+1} \left( \frac{\theta k^2}{2} \right) = 0, \ -M \leq m < 0, \quad k^2 = \frac{2\mu E}{\hbar^2}, \]
Figure: Spectrum of the infinite non-commutative well: Note that the spectrum truncates at angular momentum \(-M\) and for each value of angular momentum.
TD of a 2-D NC Fermi gas

The q-potential for the grand canonical ensemble with fixed averaged total angular momentum reads

\[ q(M, \tilde{\beta}, \tilde{\mu}, \tilde{\omega}) = \sum_{m=-M}^{\infty} \sum_{r} \log[1 + e^{-\tilde{\beta}(x_{r,m} - \tilde{\mu} - \tilde{\omega}m)}] \]

in terms of the dimensionless parameters \( \tilde{\beta} = E_0 \beta \), \( \tilde{\mu} = \mu / E_0 \) and \( \tilde{\omega} = \hbar \omega / E_0 \) with \( E_0 = \hbar^2 / (\theta m_0) \) and with \( x_{r,m} \) the zeros of the Laguerre polynomials.
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The central (dimensionless) thermodynamic quantities are

\[ N = \frac{1}{\tilde{\beta}} \frac{\partial q}{\partial \tilde{\mu}}, \quad L = \frac{1}{\hbar} \frac{\partial q}{\partial \tilde{\omega}}, \quad \frac{S}{k} = q - \tilde{\beta} \frac{\partial q}{\partial \tilde{\beta}} \]
We also define dimensionless measures of the density and pressure by \( \tilde{\rho} = \frac{N}{(2M + 1)} \) and \( \tilde{P} = \frac{q}{(\tilde{\beta}(2M + 1))} \).
TD of a 2-D NC Fermi gas (contd)

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* In the thermodynamic limit the q-potential can be written as

\[
q(M, \tilde{\beta}, \tilde{\mu}, \tilde{\omega}) = \int_{-M}^{\infty} dm \int_{x-(m)}^{x+(m)} dx \, D(x, m) \log[1 + e^{-\tilde{\beta}(x-\tilde{\mu}-\tilde{\omega}m)}]
\]

where \( D(x, m) \) is the asymptotic density of the zeros of the Laguerre polynomials

\[
D(x, m) = \frac{\sqrt{4Mx - (m-x)^2}}{2\pi x}, \quad x \in [x-(m), x+(m)]
\]
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TD of a 2-D NC Fermi gas: Low densities

- In the low density limit the density of states in the \( m \) angular momentum sector can be approximated by

\[
D(E) \approx \sqrt{\frac{2m_0 R^2 E}{\hbar^2} - m^2} \frac{2\pi E}{2}\pi E}
\]

This coincides precisely with the density of states for a commutative Fermi gas and one recovers the commutative result in the low density limit.
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TD of a 2-D NC Fermi gas: High densities

![Graph showing thermodynamics of a 2-D non-commutative Fermi gas at high densities.]
The first observation is that for the non-commutative gas there exists a maximum (critical) density at which the system is incompressible. Generally this correspond to macroscopic system sizes.
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In the high density, low temperature limit the q-potential can be computed from which the critical (maximum) density for $L = 0$ follows to be $	ilde{\rho}_c = M \nu_c$ with

$$\nu_c = \frac{(3 + 2\sqrt{3})}{12}.$$
The entropy and pressure close to the critical density can also be computed easily:

\[ S \sim \sqrt{M} \sqrt{M \nu_c - \tilde{\rho}}, \quad \tilde{P} \sim \frac{M^{5/2}}{\sqrt{M \nu_c - \tilde{\rho}}}. \]
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- As the density approaches the critical density the entropy vanishes and the pressure diverges.

- How does the entropy of a very dense gas, close to its maximum density, behave as a function of size when more particles are added and the size of the object increased, but in such a way that the density is always kept very close to the maximal density?
This turns out to be

\[ S(M_c) \sim \sqrt{2\nu_c} \sqrt{M_c} \sim R_c \]

as \( R_c^2 = \theta(2M_c + 1) \). Thus the entropy scales like the circumference, rather than the area at minimal size.
What is the thermodynamics of a 3 dimensional non-commutative Fermi gas based on the following commutation relations:

\[ [x_i, x_j] = i\theta_{i,j} \]
TD of a 3-D NC Fermi gas

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- Through an appropriate choice of coordinates it is always possible to restrict the non-commutativity to two of the spacial coordinates with the third coordinate being commutative and the q-potential can be computed as before and the following transpire:
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At high densities a cross-over to the thermodynamics of a one-dimensional Fermi gas occurs and there is no incompressibility. This seems to rule out these commutation relations, and the associated breaking of rotational symmetry, as unphysical.
Thermodynamics of a 3-D non-commutative Fermi gas (contd)

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- At high densities a cross-over to the thermodynamics of a one-dimensional Fermi gas occurs and there is no incompressibility. This seems to rule out these commutation relations, and the associated breaking of rotational symmetry, as unphysical.
- A more appropriate set of commutations relations may be those for the fuzzy sphere:

\[ [x_i, x_j] = i\theta\epsilon_{i,j,k} x_k \]
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  - There is a maximal (critical) density,
  - At the critical density the entropy vanishes,
  - At the critical density pressure diverges and the system is incompressible,
  - Close to the critical density entropy scales like the circumference rather than area of the minimal size.
In 3 dimensions the symmetry breaking commutation relations seem inappropriate.