Every known SIC has a group covariance property.

For such a SIC the projectors are labelled by the elements of a finite group $G$. For each group element $g$ there is a unitary $U_g$ such that

$$U_g \Pi_{gg'} U_g^\dagger = \Pi_{gg'}$$

Action is transitive: can get every projector starting from any single projector
The overwhelming majority of known SICs are covariant under a specific group: the Weyl-Heisenberg group.

Weyl-Heisenberg SICs exist in every dimension up to 67 (+ some more).

In prime dimensions Weyl-Heisenberg SICs are the only group covariant SICs (Zhu, J.Phys.A 43, 305305, 2010)

Suggests that the Weyl-Heisenberg group is of special importance.
Weyl-Heisenberg displacement operators in infinite dimension:

\[ D_{x,p} |x'\rangle = e^{i(px+2px')} |x' + x\rangle \]
\[
(\hbar = 2, \ x \in \mathbb{R})
\]

Weyl-Heisenberg displacement operators in finite dimension \(d\):

\[ D_{x,p} |x'\rangle = \tau^{(px+2px')} |x' + x\rangle \]
\[
(\tau = -e^{\frac{\pi i}{d}}, \ x \in \mathbb{Z}_d)
\]
Symplectic transformations (in finite or infinite dimension):

\[ U_F D_{x,p} U_F^\dagger = D_{\alpha x + \beta p, \gamma x + \delta p} \]

where

\[ F = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \begin{cases} \text{SL}(2, \mathbb{Z}_d) & d < \infty \\ \text{SL}(2, \mathbb{R}) & d = \infty \end{cases} \]

In finite dimensions the displacement operators and symplectic unitaries, taken together, generate the Clifford group.
The infinite dimensional Weyl-Heisenberg group has obvious physical importance.

It might be thought that the finite dimensional case, by contrast, is a somewhat arbitrary construct.

But if we are right to think that SIC’s are saying something of fundamental importance about the geometry of quantum state space then it would seem to follow that the discrete Weyl-Heisenberg is also of fundamental significance.
In this connection it is interesting to note that before the publication of the 1925 paper in which Born and Jordan first proposed the commutation relation

\[ [\hat{x}, \hat{p}] = i\hbar \]

Weyl suggested (in correspondence with Born) that the relation

\[ e^{ip\hat{x}} e^{ix\hat{p}} = \text{phase} \times e^{ix\hat{p}} e^{ip\hat{x}} \]

might be preferable—one of the advantages being that this form of the CCR, unlike the Born-Jordan form, applies to discrete systems as well as continuous ones.
Weyl-Heisenberg covariant POVM (finite or infinite $d$):

$$E_{x,p} = k D_{x,p} \Pi D_{x,p}^\dagger$$

Suppose we try to choose $\Pi$ in such a way as to make the $E_{x,p}$ as close to orthogonal as possible.

More precisely: suppose we try to minimize

$$\text{Tr}(E_{x,p} E_{x',p'}) = \text{Tr}(E_{x-x',p-p'} E_{0,0})$$

for $(x, p), (x', p')$ distinct
Infinite dimensional case.

It seems natural to minimize the quantities

$$\Delta x = \left( \int (x - x')^2 \text{Tr}(E_{x-x',p-p'}E_{0,0}) \, dx \, dp \right)^{1/2}$$

$$\Delta p = \left( \int (p - p')^2 \text{Tr}(E_{x-x',p-p'}E_{0,0}) \, dx \, dp \right)^{1/2}$$

one then finds that the projector $\Pi$ must be a squeezed coherent state.
Finite-dimensional case.

Due to the cyclicity and finiteness the rms measures are not very natural in this case. So instead try to minimize the quantity

$$\sum_{x,x',p,p'} \left( \text{Tr}(E_{x-x',p-p'}E_{0,0}) \right)^2$$

One then finds that the projector $\Pi$ must be a SIC fiducial.
In this sense SIC fiducials are finite dimensional analogues of squeezed coherent states.

**BUT**

The analogy only goes so far. Infinity is not a big integer. Lots of ways in which a SIC fiducial is *nothing* like a coherent state.
Interesting to look at squeezed coherent state POVMs in a little more detail.

It is still not as widely appreciated as it ought to be (naming no names) that the $Q$ function (the distribution corresponding to such a POVM) is not just a *pseudo*-probability distribution. It actually is the probability distribution corresponding to a perfectly good—indeed distinctly interesting—measurement.
Given a joint (quantum) measurement of $x$ and $p$ there is a natural way to define the measurement errors (in strictly quantum mechanical terms). The errors satisfy

$$\Delta_e x \Delta_e p \geq \frac{1}{2}$$

The quantities on the right hand side are errors not uncertainties. So this is not the relation usually described as the Heisenberg Uncertainty Relation (though it may be what Heisenberg actually had in mind when he first wrote it down).

Suppose the measurement is optimal:

\[ \Delta_e x \Delta_e p = \frac{1}{2} \]

Then

\[ \Delta_e x = \frac{\lambda}{\sqrt{2}} \quad \Delta_e p = \frac{1}{\sqrt{2} \lambda} \]

for some \( \lambda \). It turns out that for each \( \lambda \) there is a unique squeezed state POVM describing the measurement outcome.

In that sense these POVMs are canonical for joint \( x, p \) measurements.

This gives us a way of understanding the relation between classical and quantum. To go from classical to quantum simply replace

the classical phase space distribution

with the quantum $Q$ function

Question: how do we tell them apart?
Perhaps the difference is that the $Q$ function collapses?

Not so. Collapse is a generic feature of all probability distributions, classical or quantum.
But there has to be some difference. Coherent state POVMs are informationally complete (in a sense). This means that the distribution is a complete description of the quantum state used to construct it (in this case the $n=8$ energy eigenstate of an harmonic oscillator).

So the fact that it is quantum not classical has to show somehow or other.
First difference: contextuality
These 3 distributions are not different convolutions of a single underlying probability distribution (graphical illustration of contextuality)
Second difference: analyticity
Informational completeness means that

this

is somehow contained in

this

But how can that be?
Answer: the distribution is analytic in $x$ and $p$ separately, and it continues to a holomorphic function on the whole of complexified phase space.

This means that each tiny piece of the distribution fully determines the whole thing.

It also means that given the distribution for one value of $\lambda$ one can easily reconstruct the distribution for any other value.
Transformation formulae

If $\lambda' < \lambda$

$$Q_{\lambda'}(ix, p) = \frac{4\pi \lambda \lambda'}{\lambda^2 - \lambda'^2} \int e^{-\frac{\lambda \lambda'}{\lambda^2 - \lambda'^2} \left( \frac{1}{\lambda \lambda'} (x-x')^2 + \lambda \lambda' (p-p')^2 \right)} Q_{\lambda}(ix', p') \, dx' \, dp'$$

If $\lambda' > \lambda$

$$Q_{\lambda'}(x, -ip) = \frac{4\pi \lambda \lambda'}{\lambda'^2 - \lambda^2} \int e^{-\frac{\lambda \lambda'}{\lambda'^2 - \lambda^2} \left( \frac{1}{\lambda \lambda'} (x-x')^2 + \lambda \lambda' (p-p')^2 \right)} Q_{\lambda}(x', -ip') \, dx' \, dp'$$

To reduce the value of $\lambda$ continue to the imaginary $x$ axis, apply the first formula, then continue back to the real $x$ axis.

To increase the value of $\lambda$ continue to the imaginary $p$ axis, apply the second formula, then continue back to the real $p$ axis.
However, for the $Q$ function all of this should be taken with a grain of salt.

The $Q$ function (unlike a SIC) is only informationally complete in a pure mathematical sense. Practically speaking it does not give anything like a full description of the quantum state (exponential growth of errors).

Also the fine-grained information contained in the analyticity includes information about behaviour on the Planck scale and smaller—physically meaningless, therefore.
Nevertheless, I think it is relevant because it serves to illustrate the kind of thing I am looking for (but haven’t yet found) in connection with the SIC problem.

If one is only familiar with the standard $x$ space representation in terms of Lebesgue measurable functions (including, as they do, numerous highly pathological objects), the existence of a representation in terms of holomorphic phase space distributions comes as a big surprise.

I suspect that the key to the SIC problem will another equally—perhaps even more suprising—transformation.

The reason the SIC problem is hard is because we are looking at it the wrong way. Reformulate it in the right mathematical language and it is possible that SIC existence would suddenly look obvious.

More generally, a different mathematical language might be richly productive of insights into the geometry of quantum state space.
Possibilities considered so far:

1) Elliptic curves (seminar talks by Hughston and Bengtsson)

2) Modular forms (work in progress)


4) Galois Theory (to appear shortly)

(1) and (2) depend on an embedding of the *discrete* Weyl-Heisenberg group into the *continuous* one which involves holomorphic phase space distributions closely related to the $Q$ function.
Galois group: a very simple example

The complex numbers

\[ \mathbb{C} = \mathbb{R}(i) \]

obtained by adding to the reals the single generator \( i \).
Consists of all combinations of the form

\[ a + ib \]

with \( a, b \) real.

Galois group consists of the identity together with complex conjugation:

\[ g: i \rightarrow -i \]
Another simple example:

\[ F = \mathbb{Q}(\sqrt{2}) \]

the number field obtained by adding to the rationals the number \( \sqrt{2} \)
Consists of all combinations of the form

\[ a + b\sqrt{2} \]

with \( a, b \) rational.

Galois group consists of the identity together with the map

\[ g: \sqrt{2} \rightarrow -\sqrt{2} \]
A slightly less trivial example

\[ F = \mathbb{Q} \left( i, \sqrt{\sqrt{2} + 1} \right) \]

consists of all combinations of the form

\[(a_1 + ia_2) + (b_1 + ib_2)\sqrt{2} + (c_1 + ic_2)\sqrt{\sqrt{2} + 1} + (d_1 + id_2)\sqrt{2\sqrt{2} + 2}\]

with \(a_1, \ldots, d_2\) rational

Here the Galois group is non-Abelian—as is typically the case.
Galois group for a Weyl-Heisenberg SIC

Exact Weyl-Heisenberg SICs have been found in dimensions 2–16, 19, 24, 28, 31, 35, 37, 43, 48.

Expressions are in general very complicated (often several pages of print out). However the number fields all turn out to have a certain remarkable property.

In every case the field containing the components is an extension of

$$\mathbb{Q}\left(\sqrt{(d - 3)(d + 1)}\right)$$

having an Abelian Galois group (where $d$ is the dimension).
Interplay between Galois and Clifford group

Anti-unitary:

unitary + complex conjugation

Generalization: g-unitary

unitary + general Galois operation

Every (known, exact, Weyl-Heisenberg) SIC vector is an eigenvector of a group of g-unitaries. g-unitaries are typically not diagonalizable so this is a strong constraint.