Talk given at NITheP, Stellenbosch, South Africa:

Deformed Quantization and Hopf Algebras

Francesco Toppan

TEO, CBPF (MCTI)
Rio de Janeiro, Brazil

Group “Algebraic Structures in Field Theory”

Line of research on Deformed Quantization:

- 1 Concluded Ph. D. (P.G. Castro, 2010),
- 1 Concluded Master (E. Quijada, 2011),
- 1 Ph.D. orientation (R. Kullock),
- 1 Master orientation (N. Linneu).

Collaborators:

- B. Chakraborty (SN Bose, Kolkata, India) with TWAS,
- Z. Kuznetsova (UFABC),
- M. Rojas (Lavras).

ASFT papers on Deformed Quantization (so far)

- **P. G. Castro, B. Chakraborty, F. T.**,Wigner Oscillators, Twisted Hopf Algebras and Second Quantization, **JMP 49, 082106 (2008)**
- **Z. Kuznetsova, M. Rojas, F. T.**, On Supergroups with Odd Clifford Parameters and Modified Leibniz Rule Susy, **IJMPA 23, 309 (2008)**
- **B. Chakraborty, Z. Kuznetsova, F. T.**, Twist Deformation of Rotationally Invariant Quantum Mechanics, **JMP 51, 112102 (2010)**
- **P. G. Castro, B. Chakraborty, Z. Kuznetsova, F. T.**, Twist Deformations of the Susy Quantum Mechanics, **CEJP 9, 841 (2011)**
- **P. G. Castro, B. Chakraborty, R. Kullock, F. T.**, NC oscillators from a Hopf algebra twist deformation, **JMP 52, 032102 (2011)**
- **P. G. Castro, R. Kullock, F. T.**, Snyder NC-ity and Pseudo-Hermitian Hamiltonians from a Jordanian Twist, **JMP 52, 062105 (2011)**

Main results:

Introduction of a Quantization which is compatible with the physical requirement of the Hopf algebra coproduct.

Problem with the standard quantization based on creation/annihilation operators satisfying CCR and acting on a Fock space.

Solutions:

I - JMP'08: the Wigner *superquantization* (Wigner's oscillators)
E. Wigner, Do the equations of motion determine the quantum mechanical commutation relations?, PRD 1950.
Replace CCR with $osp(1|2)$ *superalgebra and the Fock space with a lowest weight representation (the lowest weight is the vacuum energy).*

II - JMP'10: the Unfolded Quantization Framework
and its developments (two JMP'11 papers) .

Main motivations on deformations

Bayen-Flato-Fronsdal-Lichnerowicz-Sternheimer from '70s:
M. Flato, *Deformation view of physical theory*, Cz. J. of Phys. 1982.

Quantum groups:
end of 70's and 80's Leningrad's school (Faddeev-Takhtajan et al.) on Inverse Scattering Methods, Jimbo-Miwa et al. in the 80's.

Key example: spin-chain Hamiltonians: XXZ $\rightarrow$ XXX. The symmetry is controlled by q, a non-dimensional parameter which measures the anisotropy: $SU(2)$ is extended to $SU_q(2)$ (notion of quantum group).

QG mathematical framework:
end of '80s: Drinfeld (Quantum groups), Woronowicz (pseudo-groups).
The Hopf algebra structure is a central concept.

Naturalness of Hopf algebra

Applied to *Fun*, functions on a group. The basic example is $(\mathbf{R}, +)$.
Coproduct Δ, $\Delta : Fun \rightarrow Fun \otimes Fun$.
The coproduct informs how to deal with functions of many variables:

$$\begin{aligned} sin(x+y) &= sin(x)cos(y) + cos(x)sin(y), \\ cos(x+y) &= cos(x)cos(y) - sin(x)sin(y), \\ exp(x+y) &= exp(x)exp(y), \\ id(x+y) &= id(x) + id(y), \qquad id(x) = x. \end{aligned}$$

$$\begin{aligned} \Delta(sin) &= sin \otimes cos + cos \otimes sin, \\ \Delta(cos) &= cos \otimes cos - sin \otimes sin, \\ \Delta(exp) &= exp \otimes exp, \\ \Delta(id) &= id \otimes \mathbf{1} + id \otimes \mathbf{1}. \end{aligned}$$

Deformations depending on a dimensional parameter

Physical applications:

Critical theories away of criticality: massive deformations of BPZ 2d CFT's minimal models.
Sine(h)-Gordon as a massive deformation of Liouville theory (Zamolodchikov, Smirnov).

Possible applications:

effective theories of Quantum gravity: quantum deviations as perturbations of classical gravity.

Constants of nature (?)

- speed of light c (special relativity)
- Planck's constant $\hbar$ (quantum regime)
- Netwon's constant G (gravitation)
- Boltzmann's constant k_B (thermodynamics)
- Hubble's constant(?) H (cosmology)

They can be used as conversion factors.

The Planck's units of length, time, energy, etc., based on $c, \hbar, G$, define the regimes where the Quantum Gravity should play a role:

$l_{Pl} \approx 10^{-33} cm$,
$t_{PL} \approx 10^{-43} sec$,
$E_{Pl} \approx 10^{19} GeV$,
...
Viewpoint: l_{Pl} as a (dimensional) deformation parameter.

Dimensional deformations

Alain Connes Noncommutative Geometry (1990 book).

NC -manifold coordinates

$$[x_a, x_b] = \frac{1}{l^2} B_{ab} + \frac{1}{l} f_{ab}^{c} x_c + R_{ab}^{cd} x_c x_d + \ldots,$$

B_{ab} constant NC, f_{ab}^{c} Lie-algebra NC, R_{ab}^{cd} Yang-Baxter NC, ...

Lukierski-Nowicki-Ruegg-Tolstoy, *kappa-Poincaré*, PLB 1991.

J. Madore, *Fuzzy sphere*, CQG 1992.

Amelino-Camelia, Ellis, Mavromatos, *Nature 1998*, test of GZK bound from gamma-ray burst (we have now Auger in function).

Amelino-Camelia (2000), DSR theories or D-special relativity ("D" originally indicated "double", while now stands for "deformed"). kappa-Poincaré is an example of DSR (far from being unique).

Still unresolved clash between QM and GR

Can it be resolved? E. Wigner, *The unreasonable effectiveness of mathematics in the natural sciences CPAM, 1960.*

Different mathematical tools:

GR $\leftrightarrow$ geometry (Riemannian manifolds).

QM $\leftrightarrow$ Operators on Hilbert spaces.

Very roughly:
string theory attempt to QG comes from QFT (QM) community,
LQG attempt to QG from GR community.

Some motivations to look at deformed QT's from Hilbert's space viewpoint:

In ordinary QFT the relation between geometry and Hilbert space is provided by the Wightman's Reconstruction Theorem.
Through the analytical continuation, an Euclidean statistical mechanics is reinterpreted as a Quantum Theory.
(Recall that the Feynman's path integral, which allows working within the Lagrangian framework, only makes sense in the Euclidean.)
What about "Deformed Quantum Theories?"
If we look at the deformation of geometry (via, e.g., star-product) we have only one side of the picture.
What about the Hamiltonian (QM-side)?
Related question: In searching for QG, which notion should we retain as more fundamental, QM or GR?
Maybe QM, because we can recover classical geometry from QM, while we cannot recover QM (Bell's inequalities) from a local classical theory.

The Unfolded Quantization

The quantization must be compatible with the physical interpretation of the coproduct.

Basic examples of the Euclidean Lie algebras $e(2)$ and $e(3)$.

Additive operators have to be assumed as "primitive elements" of the dynamical symmetry algebra.

The additivity of the eigenvalues is encoded in the undeformed coproduct:

$$\Delta(\Omega) \;\; = \;\; \Omega \otimes \mathbf{1} + \mathbf{1} \otimes \Omega$$

$(\omega_{1+2} = \omega_1 + \omega_2)$ for the operator Ω.

What about the Hamiltonian? From physical considerations we are forced to reject the Hopf algebra equivalence $H = \vec{p}^2$ $(m = \frac{1}{2})$, since (*unphyisical* coproduct rule)

$$\begin{aligned} \Delta(H) &= \Delta(\vec{p}^2) = \Delta(\vec{p}) \cdot \Delta(\vec{p}) = \\ &= \vec{p}^2 \otimes \mathbf{1} + \mathbf{1} \otimes \vec{p}^2 + 2\vec{p} \otimes \vec{p} \neq H \otimes \mathbf{1} + \mathbf{1} \otimes H. \end{aligned}$$

Correct relation (mathematically and physically) for the angular momenta $\vec{L}$:

$$\Delta(\vec{L}^2) \;=\; \Delta(\vec{L})\cdot\Delta(\vec{L}) = \vec{L}^2\otimes\mathbf{1}+\mathbf{1}\otimes\vec{L}^2+2\vec{L}\otimes\vec{L}.$$

$\vec{L}^2$ is not an additive operator, but a composite operator $(\vec{L}_{1+2})^2 = (\vec{L}_1+\vec{L}_2)^2$.

Example of $e(2)$: three generators p_1, p_2, L, satisfying the commutation relations

$$\begin{aligned} [p_1, L] &= -ip_2,\\ [p_2, L] &= ip_1,\\ [p_1, p_2] &= 0. \end{aligned}$$

$e(2)$ admits only one Casimir operator

$$\mathcal{C}\equiv\vec{p}^2 \;=\; {p_1}^2+{p_2}^2$$

(the energy E of a non-relativistic, free, two-dimensional particle with $m=\frac{1}{2}$).

The Casimir operator $\mathcal{C}$ **has to be added** to the dynamical symmetry Lie algebra. We need to enlarge $e(2)$ by defining

Unfolded dynamical Lie algebra of the harmonic oscillator

The d-dimensional Heisenberg algebra $\mathcal{H}_d$ with generators $\hbar$ (a central charge), x_i and p_i ($i = 1, 2, \ldots, d$)

$$[x_i p_j] = i\hbar\delta_{ij}, \qquad [\hbar, x_i] = [\hbar, p_i] = 0.$$

allows us to introduce the enlarged Lie algebra $\mathcal{G}_d$, containing $\mathcal{H}_d$ as a subalgebra, together with the extra generators H, K, D and $L_{i_1 \ldots i_{d-2}}$:

$$\mathcal{G}_d = \{\hbar, x_i, p_i, H, K, D, L_{i_1 i_2 \cdots i_{d-2}}\}, \; i = 1, ..., d.$$

The commutation relations within $\mathcal{G}_d$ are recovered from

$$\begin{aligned} H &= \frac{1}{2\hbar}(p_i p_i), \\ K &= \frac{1}{2\hbar}(x_i x_i), \\ D &= \frac{1}{4\hbar}(x_i p_i + p_i x_i), \\ L_{i_1 i_2 \cdots i_{d-2}} &= \frac{1}{\hbar}\epsilon_{i_1 i_2 \cdots i_{d-1} i_d} x_{i_{d-1}} p_{i_d} \end{aligned}$$

In $d = 3$ we get

$$
\begin{aligned}
[x_i, p_j] &= i\hbar\delta_{ij}, \\
[D, H] &= iH, \\
[D, K] &= -iK, \\
[K, H] &= 2iD, \\
[x_i, H] &= ip_i, \\
[x_i, D] &= \frac{i}{2}x_i, \\
[p_i, K] &= -ix_i, \\
[p_i, D] &= -\frac{i}{2}p_i, \\
[L_i, x_j] &= i\epsilon_{ijk}x_k, \\
[L_i, p_j] &= i\epsilon_{ijk}p_k, \\
[L_i, L_j] &= i\epsilon_{ijk}L_k
\end{aligned}
$$

(the remaining commutation relations are vanishing).

We introduce the Hopf algebra structure for the Universal Enveloping Lie Algebra $\mathcal{U}(\mathcal{G}_d)$.
The identifications which only hold at a Lie algebra level, but not as a Hopf algebra relation are weak equalities ("$\approx$").
In $d = 3$, we get for the third component L_z of the angular momentum the weak identification

$$\hbar L_z \quad \approx \quad xp_y - yp_x .$$

Indeed, the coproduct of the l.h.s. does not coincide with the coproduct of the r.h.s.
Similarly, the $\hbar$ generator can be identified with the **1** identity operator only weakly

$$\hbar \quad \approx \quad \mathbf{1} .$$

When representing the $\mathcal{G}_d$ generators as operators acting on a module we are dealing with the "$\approx$" weak equivalence.

Hopf algebras and Drinfel'd twist

Let $(H, \mu, \eta, \Delta, \epsilon, S)$ be a cocommutative Hopf algebra and $\mathcal{F} \in H \otimes H$ a counitary 2-cocycle:
$(\mathbf{1} \otimes \mathcal{F})(id \otimes \Delta)\mathcal{F} = (\mathcal{F} \otimes \mathbf{1})(\Delta \otimes id)\mathcal{F}$
and
$(\epsilon \otimes id)\mathcal{F} = \mathbf{1} = (id \otimes \epsilon)\mathcal{F}$.
Then $\chi = \mu(id \otimes S)\mathcal{F}$ is an invertible element of H with $\chi^{-1} = \mu(S \otimes id)\mathcal{F}^{-1}$.
We define $\Delta^{\mathcal{F}} : H \to H \otimes H$ and $S^{\mathcal{F}} : H \to H$ as $\Delta^{\mathcal{F}} = \mathcal{F}\Delta\mathcal{F}^{-1}$ and $S^{\mathcal{F}} = \chi S \chi^{-1}$.
$(H, \mu, \eta, \Delta^{\mathcal{F}}, \epsilon, S^{\mathcal{F}})$ is a Hopf algebra .
The element $\mathcal{F}$ is called a *twist*.
H is the same vector space as $H^{\mathcal{F}}$. Only the co-structures are deformed.

The abelian twist-deformed $\mathcal{U}^{\mathcal{F}}(\mathcal{G}_d)$ Hopf algebra

$\mathcal{U}(\mathcal{G}_d)$ can be deformed via the twist

$$\mathcal{F} = \exp\left(i\alpha_{ij}p_i \otimes p_j\right), \quad \alpha_{ij} = -\alpha_{ji},$$

The twist is well-defined due to the fact that the p_i momenta are among the generators of $\mathcal{G}_d$.
The twist induces a deformation ($g \mapsto g^{\mathcal{F}}$) for the $\mathcal{G}_d$ generators, with $g^{\mathcal{F}}$ belonging to $\mathcal{U}(\mathcal{G}_d)$:

$$\begin{aligned} x_i^{\mathcal{F}} &= x_i - \alpha_{ij}p_j\hbar, \\ K^{\mathcal{F}} &= K - \alpha_{ij}x_ip_j + \frac{\alpha_{jk}\alpha_{jl}}{2!}p_kp_l\hbar, \\ L^{\mathcal{F}}_{i_1i_2\cdots i_{d-2}} &= L_{i_1i_2\cdots i_{d-2}} - \epsilon_{i_1i_2\cdots i_{d-2}jk}\alpha_{jl}p_kp_l. \end{aligned}$$

The deformation of the position operators x_i corresponds to the Bopp shift.
Commutators: $[x_i^{\mathcal{F}}, x_j^{\mathcal{F}}] = i\Theta_{ij}$, where the constant operator Θ_{ij} is given by $\Theta_{ij} = 2\alpha_{ij}\hbar^2$.

Single-particle operators:

the knowledge of the deformed generators, together with their commutators and their action on a module V which possesses the structure of a Hilbert space, is sufficient to quantize the system.
Multi-particle operators: the extra-structure of the (deformed) coproduct plays a role. The deformed 2-particle operator associated with the deformed generator $g^{\mathcal{F}}$ is constructed by applying $\Delta^{\mathcal{F}}(g^{\mathcal{F}}) \in \mathcal{U}^{\mathcal{F}}(\mathcal{G}_d) \otimes \mathcal{U}^{\mathcal{F}}(\mathcal{G}_d)$ to the Hilbert space $V \otimes V$. The twist $\mathcal{F}$ applied to $V \otimes V$, corresponds to the unitary operator F. Since

$$\Delta^{\mathcal{F}}(g^{\mathcal{F}}) \;=\; \mathcal{F} \cdot \Delta(g^{\mathcal{F}}) \cdot \mathcal{F}^{-1},$$

with $\Delta(g^{\mathcal{F}})$ the undeformed coproduct, we end up that the operators $\widehat{\Delta^{\mathcal{F}}}(g^{\mathcal{F}})$, $\widehat{\Delta}(g^{\mathcal{F}})$, acting on $V \otimes V$, are unitarily equivalent:

$$\widehat{\Delta^{\mathcal{F}}}(g^{\mathcal{F}}) \;=\; F \cdot \widehat{\Delta}(g^{\mathcal{F}}) \cdot F^{-1}.$$

Viable scheme to first-quantize an abelian twist-deformed (noncommutative) quantum mechanical system:
- at first the dynamical Lie algebra $\mathcal{G}_d$.
- Next, we represent it on the Hilbert space V.
- Later we introduce the twist-deformation and realize the single-particle operators as deformed generators $g^{\mathcal{F}}$ acting on V.
- The multi-particle operators are constructed by applying the (undeformed) coproducts of $g^{\mathcal{F}}$ on the tensor space $V \otimes \ldots \otimes V$.
The choice of using the undeformed coproduct (instead of the unitarily equivalent deformed coproduct) is particularly useful since $\Delta(\mathbf{H}^{\mathcal{F}})$, i.e. the deformed 2-particle Hamiltonian, turns out to be automatically symmetric in the exchange between first and second particle.

Application: the $d = 2$ twisted oscillator

In $d = 2$ the deformation parameter α_{ij} is

$$\alpha_{12} = \epsilon_{12}\frac{\alpha}{Z}.$$

Z is a constant unit reference with the dimension of $[p^2]$, the square of the momentum, so that α is a non-dimensional parameter. Set $Z = 1$, α to belong to the fundamental domain $\alpha \in [0, +\infty]$.
$\alpha = 0$ corresponds to the undeformed 2-dimensional harmonic oscillator, the $\alpha \to +\infty$ limit is non-singular.
Construction of the Hilbert space V in terms of ordinary oscillators:

$$a_i = \frac{x_i - ip_i}{\sqrt{2}},$$

$$a_i^\dagger = \frac{x_i + ip_i}{\sqrt{2}},$$

where

$$[a_i, a_j^\dagger] = \hbar\delta_{ij}$$

a different basis:

$$b_\pm = \frac{a_x \mp i a_y}{\sqrt{2}},$$
$$b_\pm^\dagger = \frac{a_x^\dagger \pm i a_y^\dagger}{\sqrt{2}},$$

with

$$[b_\pm, b_\pm^\dagger] = \hbar.$$

Then

$$[\mathbf{H}, b_\pm] = -b_\pm,$$
$$[\mathbf{H}, b_\pm^\dagger] = b_\pm^\dagger.$$

In $d = 2$ the angular momentum L is a scalar. It satisfies the commutation relations

$$[L, b_\pm] = \mp b_\pm,$$
$$[L, b_\pm^\dagger] = \pm b_\pm^\dagger.$$

The deformed Hamiltonian $\mathbf{H}^{\mathcal{F}} \in \mathcal{U}^{\mathcal{F}}(\mathcal{G}_2)$ is

The deformed 2-particle Hamiltonian belonging to $\mathcal{U}^{\mathcal{F}}(\mathcal{G}_2) \otimes \mathcal{U}^{\mathcal{F}}(\mathcal{G}_2)$ is unambiguously fixed in terms of the coproduct.

$$\begin{aligned}\Delta(\mathbf{H}^{\mathcal{F}}) &= \mathbf{H}^{\mathcal{F}} \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{H}^{\mathcal{F}} + \\ &\alpha(y \otimes p_x + p_x \otimes y - x \otimes p_y - p_y \otimes x) + \\ &\frac{\alpha^2}{2} \sum_{i=1}^{2} (2p_i\hbar \otimes p_i + 2p_i \otimes p_i\hbar + \\ &p_i^2 \otimes \hbar + \hbar \otimes p_i^2).\end{aligned}$$

The r.h.s. is symmetric in the exchange of the first with the second particle.

Single particle spectrum

Fock space V obtained by repeatedly applying the creation operators $b_\pm^\dagger$ on the Fock vacuum $|0\rangle$ ($b_\pm|0\rangle = 0$).
We are entitled to weakly set $\hbar = 1$,
Since we are dealing with operators acting on a module, all equalities are weak equalities.
The Hamiltonian reads as

$$\mathbf{H} \;=\; \frac{1}{2}\sum_{i=\pm}\{b_i, b_i^\dagger\},$$

while a number operator N and the angular momentum L

$$\begin{aligned} N &= b_+^\dagger b_+ + b_-^\dagger b_- = N_+ + N_-, \\ L &= b_+^\dagger b_+ - b_-^\dagger b_- = N_+ - N_-. \end{aligned}$$

Since $[\mathbf{H}, L] = 0$, the $|n_+ n_-\rangle$ basis simultaneously diagonalizes both operators:

$$\begin{aligned} \mathbf{H}|n_+ n_-\rangle &= (n_+ + n_- + 1)|n_+ n_-\rangle, \\ L|n_+ n_-\rangle &= (n_+ - n_-)|n_+ n_-\rangle. \end{aligned}$$

It is convenient to reexpress the above results through the integers $n = n_+ + n_-$ and $m = n_+ - n_-$, so that

$$\begin{aligned} \mathbf{H}|nm\rangle &= (n+1)|nm\rangle, \\ L|nm\rangle &= m|nm\rangle. \end{aligned}$$

The deformed Hamiltonian, applied to the Hilbert space V, can be reproduced by the linear combination

$$\mathbf{H}^{\mathcal{F}} = \widetilde{\mathbf{H}} - \alpha L,$$

where

$$\widetilde{\mathbf{H}} = (1+\alpha^2)H + K$$

can be regarded as a renormalized undeformed Hamiltonian.

The spectrum of $\mathbf{H}^{\mathcal{F}}$ is

$$\mathbf{H}^{\mathcal{F}}|nm\rangle = \left[(\sqrt{1+\alpha^2})(n+1) - \alpha m\right]|nm\rangle,$$

for $m = -n, -n+2, \ldots, n-2, n$ and n a non-negative integer.
The vacuum is recovered for $n = 0$ (the vacuum energy is $\sqrt{1+\alpha^2}$).
For $n = 1, 2$ we have

$$\begin{aligned}
|1,1\rangle &: 2\sqrt{1+\alpha^2} - \alpha, \\
|1,-1\rangle &: 2\sqrt{1+\alpha^2} + \alpha, \\
|2,2\rangle &: 3\sqrt{1+\alpha^2} - 2\alpha, \\
|2,0\rangle &: 3\sqrt{1+\alpha^2}, \\
|2,-2\rangle &: 3\sqrt{1+\alpha^2} + 2\alpha.
\end{aligned}$$

This spectrum coincides with the one computed in Kijanka et l. (2004) and Scholtz et al. (2009).

In the limit for $\alpha \to +\infty$, the normalized Hamiltonian

$$\mathbf{H}_N^{\mathcal{F}} = \frac{1}{\sqrt{1+\alpha^2}}\mathbf{H}^{\mathcal{F}}$$

is well-defined and coincides with the identity operator $\mathbf{1}$ acting on the reduced Hilbert space $V' \subset V$ spanned by the vectors $|n,n\rangle$. The remaining vectors decouple from the theory because their energy gap with respect to the degenerate vacuum tends to infinity.

The multi-particle Hamiltonian

(beyond Kijacka and Scholtz)

The deformed two-particle Hamiltonian acting on $V \otimes V$ is recovered by computing, at first, the (undeformed) coproduct of the deformed Hamiltonian $\mathbf{H}^{\mathcal{F}}$.

$$\begin{aligned} \Delta(\mathbf{H}^{\mathcal{F}}) &= \mathbf{H}^{\mathcal{F}} \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{H}^{\mathcal{F}} + \\ & \alpha(y \otimes p_x + p_x \otimes y - x \otimes p_y - p_y \otimes x) \\ & + \frac{\alpha^2}{2} \sum_{i=1}^{2} (2p_i \hbar \otimes p_i + 2p_i \otimes p_i \hbar + \\ & p_i^2 \otimes \hbar + \hbar \otimes p_i^2). \end{aligned}$$

Symmetric under particle-exchange

No longer additive due to α-terms.

Even if no longer additive, the coassociativity of the coproduct guarantees in any case the associativity of the deformed Hamiltonian:

$$(id \otimes \Delta)\Delta(\mathbf{H}^{\mathcal{F}}) = (\Delta \otimes id)\Delta(\mathbf{H}^{\mathcal{F}}) \equiv \Delta_{(2)}(\mathbf{H}^{\mathcal{F}}).$$

$$
\begin{aligned}
\Delta_{(2)}(\mathbf{H}^{\mathcal{F}}) &= \mathbf{H}^{\mathcal{F}}\otimes\mathbf{1}\otimes\mathbf{1}+\mathbf{1}\otimes\mathbf{H}^{\mathcal{F}}\otimes\mathbf{1}+\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{H}^{\mathcal{F}}\\
&+ \alpha(\mathbf{1}\otimes y\otimes p_x + y\otimes\mathbf{1}\otimes p_x + y\otimes p_x\otimes\mathbf{1})\\
&+ \alpha(\mathbf{1}\otimes p_x\otimes y + p_x\otimes\mathbf{1}\otimes y + p_x\otimes y\otimes\mathbf{1})\\
&- \alpha(\mathbf{1}\otimes x\otimes p_y + x\otimes\mathbf{1}\otimes p_y + x\otimes p_y\otimes\mathbf{1})\\
&- \alpha(\mathbf{1}\otimes p_y\otimes x + p_y\otimes\mathbf{1}\otimes x + p_y\otimes x\otimes\mathbf{1})\\
&+ \alpha^2\sum_{i=1}^{2}[\mathbf{1}\otimes p_i\hbar\otimes p_i + p_i\hbar\otimes p_i\otimes\mathbf{1} + p_i\hbar\otimes p_i\otimes\mathbf{1}\\
&+ \mathbf{1}\otimes p_i\otimes p_i\hbar + p_i\otimes p_i\hbar\otimes\mathbf{1} + p_i\otimes p_i\hbar\otimes\mathbf{1}\\
&+ \hbar\otimes p_i\otimes p_i + p_i\otimes p_i\otimes\hbar + p_i\otimes p_i\otimes\hbar\\
&+ \frac{1}{2}(\mathbf{1}\otimes\hbar\otimes p_i^2 + \hbar\otimes p_i^2\otimes\mathbf{1} + \hbar\otimes p_i^2\otimes\mathbf{1}\\
&+ \mathbf{1}\otimes p_i^2\otimes\hbar + p_i^2\otimes\hbar\otimes\mathbf{1} + p_i^2\otimes\hbar\otimes\mathbf{1})].
\end{aligned}
$$

The deformed two-particle energy $E_{12}^{\mathcal{F}}$ is $E_{12}^{\mathcal{F}} = E_1^{\mathcal{F}} + E_2^{\mathcal{F}} + \Omega_{12}$, where $E_i^{\mathcal{F}}$ $(i = 1, 2)$ are the single-particle energies and Ω_{12} is an effective interaction term.

We have at least two possible interpretations for the above results. Either we regard Ω_{12} as an interaction or we regard it as describing a system of free (albeit deformed) particles, with $\Omega_{12} \neq 0$ as a measure of deformation.

The associativity is expressed by the three-particle formula

$$E^{\mathcal{F}}_{123} \equiv E^{\mathcal{F}}_{(12)3} = E^{\mathcal{F}}_{1(23)} = \\ = E^{\mathcal{F}}_{1} + E^{\mathcal{F}}_{2} + E^{\mathcal{F}}_{3} + \Omega_{12} + \Omega_{23} + \Omega_{31} + \Omega_{123}$$

with Ω_{123} recovered from the Ω_{ij}'s.

It should be stressed the crucial role of the coproduct in unambiguously determine the "interacting term" Ω_{12}.

The above formulas are equalities in the tensor products of the Universal Enveloping Lie algebras $\mathcal{U}^{\mathcal{F}}(\mathcal{G}_2) \otimes \mathcal{U}^{\mathcal{F}}(\mathcal{G}_2)$ and $\mathcal{U}^{\mathcal{F}}(\mathcal{G}_2) \otimes \mathcal{U}^{\mathcal{F}}(\mathcal{G}_2) \otimes \mathcal{U}^{\mathcal{F}}(\mathcal{G}_2)$. We specialize them now as operator equalities acting on the $V \otimes \ldots \otimes V$ multi-particle Hilbert space ($\hbar$ is mapped into the identity operator).

Comment: detecting deformations through multiparticle operators.

An examination of this spectrum shows a very important feature, namely that it can be reproduced by a linear combination of the undeformed generators (in $d = 2$ $\mathbf{H}^{\mathcal{F}}$ is reconstructed in terms of $\mathbf{H}$ and L). This observation brings an important consequence.
A measurement of the system which only involves single-particle observables is not able to detect whether the system is truly deformed or not. The deformation we are dealing with (the abelian Drinfel'd twist) only makes itself manifest in the multi-particle sector.
The measurement of the two-particle observables is required (and sufficient) to detect whether the system is deformed or not.

Rotational invariance in $d = 2$

In $d = 2$ the deformed oscillator maintains the $so(2)$ rotational invariance.

The L generator of the rotation on the xy plane, deformed under the twist according to

$$L^{\mathcal{F}} = L - \alpha(p_x^2 + p_y^2),$$

is no longer a rotation generator in the deformed case since

$$[L^{\mathcal{F}}, x_i^{\mathcal{F}}] \;=\; i(\epsilon_{ij}x_j^{\mathcal{F}} + 2\alpha p_i\hbar).$$

and does not commute with the deformed hamiltonian $\mathbf{H}^{\mathcal{F}}$.

On the other hand, there exists in the $\mathcal{U}^{\mathcal{F}}(\mathcal{G}_2)$ Universal Enveloping Algebra a generator possessing the above properties: L itself.

$$[L, x_i^{\mathcal{F}}] \;=\; i\epsilon_{ij}x_j^{\mathcal{F}},$$

$$[L, \mathbf{H}^{\mathcal{F}}] \;=\; 0.$$

Rotational invariance is maintained in the multi-particle sector:

$$\left[\Delta(\mathbf{H}^{\mathcal{F}}), \Delta(L)\right] = 0.$$

In the twist case $so(3)$ is broken to $so(2)$. Explicit computation for the deformed angular momentum

$$L_i^{\mathcal{F}} = L_i + \alpha p_i p_z - \alpha_i p_j p_j.$$

They are not be generators of rotational symmetry, since

$$[L_i^{\mathcal{F}}, x_j^{\mathcal{F}}] = i\epsilon_{ijk} x_k^{\mathcal{F}} - 2i\hbar(\delta_{ij}\alpha p_z - \alpha_i p_j).$$

Performing the computation with the L_i's,

$$[L_i, x_j^{\mathcal{F}}] = i\epsilon_{ijk} x_k^{\mathcal{F}} - i\hbar(\delta_{ij}\alpha p_z - p_i \alpha_j),$$

the second term of this expression vanishes only for $i = 3$.
$[\mathbf{H}^{\mathcal{F}}, L_i]$ also vanishes only for $i = 3$.
L_z is a generator of rotational symmetry, while L_x and L_y are not.

Noether's symmetries: twisted rotations and twisted commutators

For ordinary commutators:

$$[{x_i}^{\mathcal{F}}, {x_j}^{\mathcal{F}}] \quad = \quad i\theta_{ij}.$$

In the "opposite" direction, the $\mathcal{F}$-commutator of the ordinary coordinates produces

$$[x_i, x_j]_{\mathcal{F}} \quad = \quad -\frac{1}{2} i\theta_{ij}.$$

The $\mathcal{F}$-commutator among twisted space coordinates is vanishing

$$[{x_i}^{\mathcal{F}}, {x_j}^{\mathcal{F}}]_{\mathcal{F}} \quad = \quad 0.$$

Therefore the original $su(2)$ rotational algebra is recovered in terms of the $\mathcal{F}$-commutator of the twisted angular momentum:

$$[L_i{}^{\mathcal{F}}, L_j{}^{\mathcal{F}}]_{\mathcal{F}} = i\epsilon_{ijk} L_k{}^{\mathcal{F}}.$$

The rotational symmetry is preserved by the twist-deformation and, since,

$$\begin{aligned} [x_i^{\mathcal{F}}, L_j^{\mathcal{F}}]_{\mathcal{F}} &= i\epsilon_{ijk} x_k^{\mathcal{F}}, \\ [p_i^{\mathcal{F}}, L_j^{\mathcal{F}}]_{\mathcal{F}} &= [p_i, L_j^{\mathcal{F}}]_{\mathcal{F}} = i\epsilon_{ijk} p_k^{\mathcal{F}}, \end{aligned}$$

both $x_i^{\mathcal{F}}$ and p_i have vectorial transformation properties under the deformed brackets.
On the other hand. What happens to operators which are rotationally invariant in the undeformed case? Do they keep the rotational invariant property even in the deformed case or otherwise acquire an anomalous term which disappears in the limit $\vec{\rho} \to 0$?

The answer can be given by checking the commutation relations

$$[L_i^{\mathcal{F}}, B^\sharp]_{\mathcal{F}}$$

for an operator $B^\sharp$ belonging to the Universal Enveloping Algebra of a Lie algebra containing the Euclidean algebra $e(3)$ as a subalgebra and such that $B^\sharp$ is expanded in $\vec{\rho}$ Taylor series:

$$B^\sharp = B_0 + B_1 + B_2 + \ldots ,$$

with B_k k-linear in $\vec{\rho}$. Here $B_0 \equiv B$ denotes the undeformed limit for $\vec{\rho} \to 0$ of $B^\sharp$ (we can therefore say that the operator $B^\sharp$ is the deformation of B).
The rotational invariance in the undeformed limit requires that the following relation involving ordinary commutators and angular momentum operators has to be satisfied

$$[L_i, B_0] = 0.$$

We get the recursion relations:

$$[L_i^{\mathcal{F}}, B^\sharp]_{\mathcal{F}} \;=\; [L_i - K_i, B^\sharp] + M_{ik}[p_k, B^\sharp],$$

with M_{ik} given by $M_{ik} = 2\rho_k p_i - 2\rho_i p_k$.
The result is that a rotationally invariant operator B such that

$$[L_i, B] \;=\; 0.$$

can develop, under deformation, an anomaly A_i which is expressed through

$$[L_i^{\mathcal{F}}, B^\sharp]_{\mathcal{F}} \;=\; A_i.$$

Example (Coulomb potential):

$$\left(\frac{1}{r}\right)^\sharp \;=\; \frac{1}{r} - \hbar\epsilon_{ljk}\rho_l p_j x_k \frac{1}{r^3} + O(\hbar^2),$$

satisfies the anomalous twist-deformed commutator with minimal anomaly:

$$\left[L_i^{\mathcal{F}}, \left(\frac{1}{r}\right)^\sharp\right]_{\mathcal{F}} \;=\; \hbar^2\left(\frac{\rho_i}{r^3} - 3x_i\frac{\vec{\rho}\vec{x}}{r^5}\right) + O(\hbar^3).$$

The Jordanian twist of $\mathcal{U}(sl(2))$

$sl(2)$ (conformal) subalgebra within $\mathcal{G}_d$, expressed by K, D and H:

$$\begin{aligned}[D,H] &= iH\\ [D,K] &= -iK\\ [K,H] &= 2iD.\end{aligned}$$

The Jordanian twist is $\mathcal{F} = \exp\left(-iD\otimes\sigma\right)$, where $\sigma = \ln(\mathbf{1}+\xi H)$. The parameter ξ is dimensional and is taken as a real, positive number. The twist induces the deformation $g \mapsto g^{\mathcal{F}}$ on the generators of $\mathcal{G}_d$, with $g^{\mathcal{F}} \in \mathcal{U}(\mathcal{G}_d)$. Explicitly,

$$\begin{aligned}x_i^{\mathcal{F}} &= x_i e^{\frac{\sigma}{2}}\\ p_i^{\mathcal{F}} &= p_i e^{-\frac{\sigma}{2}}\\ H^{\mathcal{F}} &= He^{-\sigma}\\ K^{\mathcal{F}} &= Ke^{\sigma},\end{aligned}$$

the others remaining undeformed.

We recover the Snyder NC $[x_i^{\mathcal{F}}, x_j^{\mathcal{F}}] = -\frac{i\xi}{2}(x_i^{\mathcal{F}} p_j^{\mathcal{F}} - x_j^{\mathcal{F}} p_i^{\mathcal{F}})$ and

$$
\begin{aligned}
[x_i^{\mathcal{F}}, p_j^{\mathcal{F}}] &= i\hbar\delta_{ij} + \frac{i\xi}{2} p_i^{\mathcal{F}} p_j^{\mathcal{F}} \\
[x_i^{\mathcal{F}}, D^{\mathcal{F}}] &= \frac{i}{2}(x_i^{\mathcal{F}} - \xi x_i^{\mathcal{F}} H^{\mathcal{F}}) \\
[x_i^{\mathcal{F}}, H^{\mathcal{F}}] &= i p_i^{\mathcal{F}} (\mathbf{1} - \xi H^{\mathcal{F}}) \\
[x_i^{\mathcal{F}}, K^{\mathcal{F}}] &= -\frac{\xi}{2} x_i^{\mathcal{F}} \left(\mathbf{1} + \frac{\xi}{2} H^{\mathcal{F}}\right) + i\xi(K^{\mathcal{F}} p_i^{\mathcal{F}} + D^{\mathcal{F}} x_i^{\mathcal{F}}) \\
[p_i^{\mathcal{F}}, D^{\mathcal{F}}] &= -i p_i^{\mathcal{F}} \left(\mathbf{1} - \frac{\xi}{2} H^{\mathcal{F}}\right) \\
[p_i^{\mathcal{F}}, K^{\mathcal{F}}] &= -i(x_i^{\mathcal{F}} + \xi p_i^{\mathcal{F}} D^{\mathcal{F}}) + \frac{\xi^2}{4} p_i^{\mathcal{F}} H^{\mathcal{F}} \\
[D^{\mathcal{F}}, H^{\mathcal{F}}] &= i H^{\mathcal{F}} (\mathbf{1} - \xi H^{\mathcal{F}}) \\
[D^{\mathcal{F}}, K^{\mathcal{F}}] &= -i K^{\mathcal{F}} (\mathbf{1} - \xi H^{\mathcal{F}}) \\
[K^{\mathcal{F}}, H^{\mathcal{F}}] &= 2i D^{\mathcal{F}} (\mathbf{1} + \xi H^{\mathcal{F}}) + 2\xi H^{\mathcal{F}} - 2\xi^2 (H^{\mathcal{F}})^2.
\end{aligned}
$$

Pseudo-hermitian Hamiltonians.

For the harmonic oscillator the undeformed Hamiltonian is $\mathbf{H} = H + K$. The deformed Hamiltonian

$$\mathbf{H}^{\mathcal{F}} = H^{\mathcal{F}} + K^{\mathcal{F}} = He^{-\sigma} + Ke^{\sigma}$$

is η-pseudo-Hermitian: $\mathbf{H}^{\mathcal{F}\dagger} = \eta \mathbf{H}^{\mathcal{F}} \eta^{-1}$, where $\eta = e^{\sigma} = \mathbf{1} + \xi H$.
Mostafazadeh construction: the Hamiltonian becomes self-adjoint under the η-deformed inner product

$$<< \psi, \phi >> = < \psi, \eta \phi > .$$

As a vector space the Hilbert space $\tilde{\mathcal{H}}$ endowed with the η-deformed inner product is isomorphic to the original one, $\mathcal{H}$, as a Hilbert space this is not true. If η is positive definite, the new inner product will also be so.

All observables on $\tilde{\mathcal{H}}$ can be mapped back onto $\mathcal{H}$ where the inner product is the usual one, through the non-local transformation

$$\mathbf{H}^{\mathcal{F}} \mapsto \mathbf{H}^{\mathcal{F}}_{\rho} \;=\; \rho \mathbf{H}^{\mathcal{F}} \rho^{-\mathbf{1}}.$$

with $\rho = \exp \frac{1}{2}\sigma$.
The new Hamiltonian, given by

$$\mathbf{H}^{\mathcal{F}}_{\rho} \;=\; \left(1 - \frac{\xi^2}{4}\right) H^{\mathcal{F}} + K^{\mathcal{F}} + i\xi D,$$

is explicitly Hermitian since $K^{\mathcal{F}\dagger} = K^{\mathcal{F}} + 2i\xi D$.
The transformation ρ is called a pseudo-canonical transformation. The systems described by $\mathbf{H}^{\mathcal{F}}$ in $\tilde{\mathcal{H}}$ and $\mathbf{H}^{\mathcal{F}}_{\rho}$ in $\mathcal{H}$ are physically equivalent.

Just like the abelian twist, the undeformed and deformed coproducts are unitarily equivalent for all *n*-particle states. We are therefore entitled to use the undeformed coproduct $\Delta(\mathbf{H}^{\mathcal{F}})$ has the advantage of exhibiting manifest symmetry under particle exchange.
For the 2-particle case we have

$$
\begin{aligned}
\Delta(\mathbf{H}^{\mathcal{F}}) &= Ke^{\sigma}\otimes e^{\sigma}+e^{\sigma}\otimes Ke^{\sigma} - \\
&\quad -\xi^{2}(KH\otimes H+H\otimes KH)+ \\
&\quad +\sum_{n=1}^{\infty}(-\xi)^{n-1}\sum_{k=0}^{n}\binom{n}{k}H^{k}\otimes H^{n-k}
\end{aligned}
$$

It can be written as $\mathbf{H}^{\mathcal{F}}{}_{12}=\mathbf{H}^{\mathcal{F}}{}_{1}+\mathbf{H}^{\mathcal{F}}{}_{2}+\Omega_{12}$.
Regarding the three-particle states, the energy will still be associative.
This comes from the coassociativity of the coproduct.

From deformed Hopf algebras to deformed entropy?

Can statistics deformations (e.g. Tsallis entropy) be recovered from the mathematical deformation theory of Hopf algebras?

Promising relation: 1975 Takahashi-Umezawa Thermofield Dynamics.

In TFD the statistical average $\overline{\Omega}$ is recovered from a V.E.V.:

$$\overline{\Omega} = < U|\Omega|U > .$$

The Hilbert space is doubled. The physical Hilbert space is supplemented by a mirror copy. There is a natural Hopf algebra structure in TFD.

Further developments

Work in progress: from QM to DSR theories.

Application to deformed one-loop QFT's based on background split methods: classical solutions plus (deformed) quantum fluctuation. To be reported ....

That's all. Thanks for the attention.