An invitation to

Sandpiles

David Perkinson
Reed College, Portland, Oregon
Reed students/collaborators

Natalie Durgin (HMC)
Laura Florescu
Bryan Head
Sam Hopkins
Dani Morar

Jacob Perlman
Nick Salter
Seth Terashima
John Wilmes
Tianyuan Xu
Sandpiles

**Physics:**  self-organized criticality  Bak, Tang, Weisenfeld, Dhar

**Probability:**  Laplacian growth  (Levine, Peres, Propp, Sheffield, Wilson, ...)

**Riemann surfaces:**  Divisors  Baker, Norine, Musiker, ...

**Combinatorics:**  parking functions  Stanley, Postnikov
hyperplane arrangements  David Klivans, Martin, P.
matroids / Tutte  Merino
tilings  P. et al

**Combinatorial algebraic geometry:**  lattice ideals  Cori, Rossin, Salvy, Sturmfels, Mayr, Wilmes, P.
syzygies  


Outline

- sandpile group
- BTW
- sandpiles and the Laplacian
- trees
- tilings
  - alg. geo. version 1
  - alg. geo. version 2
    \[ \text{versions 1 + 2} \]
- special bonus topic
Abelian Sandpile Model

A configuration of sand on a graph
Firing an unstable vertex
Sandpile Monoid

\[ G: \text{ graph w/ sink } s \]
\[ a, b: \text{ stable configurations of sand on } \Gamma \]
\[ \text{stable addition: } a \odot b = \text{ add and stabilize} \]

Order of firing does not matter
Recurrent Configurations

Def. A configuration $c$ on $G$ is recurrent if it is stable and given any configuration $a$, $\exists$ a config. $b$ s.t. $a \otimes b = c$.

$c = a \otimes \star$?

arbitrary
The Sandpile Group

Thm. The set of recurrent configurations on \((G, s)\) forms a group, \(S(G)\), under \(\otimes\).
Sandpile model on a grid graph
BTW experiment

1. Start with some recurrent state $c$ on an $n \times n$ grid.
BTW experiment

1. Start with some recurrent state $c$ on an $n \times n$ grid.
2. Add one grain of sand to a random vertex. Stabilize and record the number of vertex firings. 

The size of the avalanche.
BTW experiment

1. Start with some recurrent state c on an nxn grid.
2. Add one grain of sand to a random vertex.
   Stabilize and record the number of vertex firings.
3. Repeat step 2 many times.
4. Let $D(N) =$ # of avalanches of size $N$.
5. Plot $D(N)$ against $N$ on log-log paper.
Figure 2. Empirical cumulative distribution functions giving the average numbers of events per year with magnitude equaling or exceeding the value on the abscissa. Curves are given for the 10 states identified in Table 2.
Self-Organized Criticality

Bak, Tang, Wiesenfeld (1987)

* system naturally evolving to a barely stable state

* instabilities obeying a power law
(reduced) Laplacian: \( \Delta \)

\[
\begin{bmatrix}
1 & 2 & 3 \\
3 & -1 & -1 \\
-1 & 3 & -1 \\
-1 & -1 & 2
\end{bmatrix}
= 
\begin{bmatrix}
3 & 0 & 0 \\
0 & 3 & 0 \\
0 & 0 & 2
\end{bmatrix}
- 
\begin{bmatrix}
1 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0
\end{bmatrix}
\]

\( \Delta = \text{degree} - \text{adjacency} \)
\[
\Delta = \begin{bmatrix}
3 & -1 & -1 \\
-1 & 3 & -1 \\
-1 & -1 & 2
\end{bmatrix}
\]
Thm. \[ \Sigma(G) \approx \frac{\mathbb{Z}^n}{\text{image}(\Delta)} \]

Example

\[ \frac{\mathbb{Z}^3}{\text{image}\left(\begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 2 \end{bmatrix}\right)} \approx \frac{\mathbb{Z}}{8\mathbb{Z}} \]
Thm. \[ S(G) \cong \frac{\mathbb{Z}^n}{\text{image}(\Delta)} \]

Cor. \[ \# S(G) = \det(\Delta) \]

\[ = \# \text{spanning trees of } G \]

\[ \uparrow \]

matrix tree theorem
# $S(G) = \# \text{trees of } G$

det $\Delta = \begin{vmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 2 \end{vmatrix} = 8$
Rotor - Routers (Levine, Propp 2008)

New firing rule
Rotor - Routers
Domino Tilings
36 domino tilings of 4x4 grid
TILINGS

Thm. (Kasteleyn; Temperly and Fisher, 1961)
The number of domino tilings of an $M \times N$ grid is

$$\prod_{1 \leq m \leq M} \prod_{1 \leq n \leq N} \left( 4 \cos^2 \left( \frac{m \pi}{M+1} \right) + 4 \cos^2 \left( \frac{n \pi}{N+1} \right) \right)^{\frac{1}{4}}.$$ 

$M = N = 4 : \quad \left[ 4 \left( \frac{\sqrt{5}+1}{4} \right)^2 + 4 \left( \frac{\sqrt{5}-1}{4} \right)^2 \right]^{\frac{1}{4}} \left[ 4 \left( \frac{\sqrt{5}+1}{4} \right)^2 + 4 \left( \frac{\sqrt{5}-1}{4} \right)^2 \right]^{\frac{1}{4}} \cdots$
number of recurrents on a 4x4 grid

= 557,568,000

number of symmetric recurrents =

\[ \psi \]
36 symmetric recurrences on a 4x4 grid

- = 3
- = 2
- = 1
- = 0
Thm. (P., Florescu, Moran, Salter, Xu)

The following are equal:

1. the number of domino tilings of a $2M \times 2N$ grid;

2. the number of symmetric recurrences on a $2M \times 2N$ grid;

3. \[
\prod_{k=1}^{N} U_{2M} \left( i \cos \left( \frac{k \pi}{2N+1} \right) \right); 
\]

4. \[
\prod_{1 \leq m \leq M \atop 1 \leq n \leq N} \left( 4 \cos^2 \left( \frac{m \pi}{2M+1} \right) + 4 \cos^2 \left( \frac{n \pi}{2N+1} \right) \right) \left[ K; T+F \right]. 
\]
domino tiling \quad \leftrightarrow \quad \text{perfect matching}
Trees and Matchings (Kenyon, Propp, Wilson, 2000)

$G$

sink

$\text{KPW}$
Trees on $G$ $\leftrightarrow$ Perfect matchings on $G^*$

1. Start with $G$.
2. Overlay dual graph.

Will now describe $G^*$.
$G \cup G^\perp$

3. Remove sinks.
Trees on $G$ $\leftrightarrow$ perfect matchings on $G^*$
KPW
Sandpiles and Algebraic Geometry
Algebraic curve
(= Riemann surfaces)

\[ f(x,y) = 0 \]
Algebraic curve
(= Riemann surface)

$f(x, y) = 0$

\[ \{(\log |x|, \log |y|) : f(xy) = 0\} \]
Algebraic curve
(= Riemann surfaces)

$f(x,y) = 0$

\( \{(\log |x|, \log |y|) : f(x,y) = 0\} \)

Tropical curves

metrized graphs
Algebraic curve
(: Riemann surfaces)

\[ f(x,y) = 0 \]

\[ \{(\log|1x|, \log|1y|) : f(xy) = 0\} \]

Amoebas

Tropical curves

Metrized graphs

Graphs
\[ y^3 - x^2 s = 0 \]
\[ x^2 - y^2 z = 0 \]
\[ z^3 - x y s = 0 \]
\[ y z^2 - s^2 = 0 \]
Set $s=1$:

\[ x^2 - yz = 0, \quad y^3 - xz = 0, \quad z^3 - xy = 0, \quad yz - 1 = 0 \]

Solutions: \[ (\omega = e^{\frac{2\pi i}{8}}) \]

\[ (1,1,1), (-1,\omega,\omega), (1,\omega^{-1},\omega^2), \ldots, (-1,\omega^{-7},\omega^7) \]
Thm.

\[ \text{Solutions} \subseteq S^1 \times \cdots \times S^1 = C^n \]

- finite abelian group \( \cong \) Sandpile group
- Solutions = orbit of a faithful representation of the sandpile group

\( n = \# \text{vertices} - 1 \)
\[ I^0 = \langle x^2 - yz, y^3 - xz, z^3 - xy, yz - s \rangle \subseteq \mathbb{C}[x, y, z] \]

\textbf{Homogenization:}

\[ I = \langle x^2 - yz, y^3 - xzs, z^3 - yzs, yz - s^2, xz^2 - y^2s, xy^2 - z^2s \rangle \]

\[ \subseteq \mathbb{C}[x, y, z, s] \]
$$I = \langle x^2 - yz, \ y^3 - xzs, \ z^3 - yxs, \ yz - s^2, \ xz^2 - y^2s, \ xy^2 - z^2s \rangle$$
• The number of generators of $I$ is the number of well-connected 2-partitions of $\Gamma$.

• The degrees of the generators give the number of edges in the corresponding cut-set.
Example

\[ C[x, y, z]/\langle x^2yz, y^3xz, z^3-xy, yz-1 \rangle \]

= \text{Span}_C \{ 1, x, y, z, xy, xz, y^2, z^2 \}

Exponent vectors: 000, 100, 010, 001, 110, 101, 020, 002

\(c_{\text{max}}\) - exp. vectors: 122, 022, 112, 121, 012, 021, 102, 120
Example

\[ \mathbb{C}[x,y,z]/\langle x^2-yz, y^3-xz, z^3-xy, yz-1 \rangle \]

\[ = \text{Span}_\mathbb{C} \{ 1, x, y, z, xy, xz, y^2, z^2 \} \]

Hilbert function for \( I^0 \): \( H(0) = 1 \), \( H(1) = 3 \), \( H(2) = 4 \)

Thm. (Merino, 1999)

\[ T(1,y) = 1 + 3y + 4y^2 \]
$T_G(x, y) = x + 2x^2 + x^3 + (1+2x)y + y^2$
\[ S = \mathbb{C}[x, y, z, s] \]

(Minimal resolution) of the ideal I:

\[ 0 \leftarrow S_{I} \leftarrow S \leftarrow S^{6} \leftarrow S^{9} \leftarrow S^{4} \leftarrow 0 \]

Betti numbers: 1, 6, 9, 4

\[ \beta_0 \beta_1 \beta_2 \beta_3 \]

"length" of resolution = 3

= # vertices - 1
Divisors

sandpile group: \( S(G) = \frac{\mathbb{Z}^n}{\text{image } (\Delta)} \)

Picard group: \( \text{Pic}(G) = \frac{\mathbb{Z}^{n+1}}{\text{image } (\Delta_{\text{full}})} \cong \mathbb{Z} \oplus S(G) \)

divisors: \( D \in \mathbb{Z}^{n+1} = \mathbb{Z}V \)

Sandpile game with no sink, all vertices may fire
Linear equivalence of divisors

\[ [D] = [D'] \in \text{Pic}(G) \]
Riemann–Roch

Thm. (Baker, Norine 2007)

\[ r(D) - r(K-D) = \deg D + 1 - g \]

\[ \text{genus} = \# E - \# V + 1 \]

\[ \deg(\sum a_v v) = \sum a_v \]

complete linear system: \( |D| = \{ E \in \mathbb{Z}V : E \sim D \text{ and } E \geq 0 \} \)

= all effective divisors linearly equivalent to \( D \).
\[ 0 \leftarrow S_{\frac{1}{4}} \leftarrow S \leftarrow \bigoplus_{i=1}^{6} S(-D_{0i}) \leftarrow \bigoplus_{i=1}^{9} S(-D_{1i}) \leftarrow \bigoplus_{i=1}^{4} S(-D_{2i}) \leftarrow 0 \]

\[
\begin{array}{cccc}
0 & 1 & 1 & 0 \\
2 & 0 & 0 & 0 \\
0 & 0 & 3 & 0 \\
1 & 0 & 2 & 0 \\
0 & 3 & 0 & 0 \\
1 & 2 & 0 & 0
\end{array}
\]

\[
D = \begin{array}{cccc}
0 & 1 & 2 & 1 \\
0 & 2 & 1 & 1 \\
1 & 2 & 0 & 1 \times 2 \\
1 & 1 & 2 & 0 \\
1 & 0 & 2 & 1 \times 2 \\
1 & 2 & 1 & 0 \\
0 & 2 & 2 & 0
\end{array}
\]

\[ |D| = \{ 1021, 2200, 0202, 0310 \} \]

\[ H_1(\Delta_D) = \mathbb{Z}^2 \]
Computation

wire \ldots 1 1 1 0 2 1 1 1 1 1 1 \ldots

(\textit{Goles, Margenstern '96})
Encoding of information

1

0
NOT-gate

0 →

1 0 2 1 1 1 1 1 1 1 1 1 1 1 1 1

→

1 1 1 1 1 1 1 0 2 1 1 1 1 1 1 1

→
OR-gate
Abelian networks

“In many applications, especially in computer science, one considers such networks where the speed of individual processors is unknown, and where the final state and outputs generated should not depend on these speeds. Thus it is essential to construct the network so that the order at which messages arrive to the processors is immaterial.”

- Deepak Dhar (1998)
"Abelian networks I: Foundations and examples"

Ben Bond, Lionel Levine (draft, November 2011)
Thanks!

Resources

- Sage Thematic Tutorial on sandpiles, www.sagemath.org
- AIMS homepage — Resources — Class notes — Sandpiles
- "Chip-firing and rotor-routing on directed graphs" Holroyd, Levin, Mészáros, Peres, Propp, Wilson
- "Primer for the algebraic geometry of sandpiles" P., Riehman, Wilmes
- www.reed.edu/~davidp

< arXiv, 45 pages